EFFECT OF A SPHERICAL SOURCE IN AN ELASTIC MEDIUM UNDER INHOMOGENEOUS BOUNDARY CONDITIONS

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The problem of the vibrations of an elastic half-space under the effect of an internal source of the expansion-compression type was solved in [1]. The steady motion of an unlimited elastic medium under the effect of a vibration source with inhomogeneous boundary conditions was considered in [2], where mainly the dependence of the amplitude of the displacement of the frequency of the boundary load was investigated. A method is indicated in this paper for obtaining the displacement and stress fields in laminar media with plane-parallel interfaces if the boundary conditions are given analogously to [2]. For this, the potentials are calculated by separation of variables in a spherical coordinate system, and are then written in a cylindrical system. As an illustration, theoretical oscillograms of the displacement on the surface of a half-space under the effect of a "thrust" source are presented. An asymptotic formula is obtained for the displacements in the neighborhood of the wave fronts under the effect of the vibration source.

1. In a spherical coordinate system (R, θ, χ) we consider a class of sources that are symmetric relative to a certain line. We superpose this line on the z axis of a cylindrical (r, χ, z) coordinate system (Fig. 1). The $z = 0$ plane separates the space into parts with different physical properties or is the free surface of a half-space.

On a sphere of radius R_0 with center at the point (0, 0, h) of the cylindrical coordinate system, let there be given a certain displacement distribution

$$u_R|_{R=R_0} = U_R(\theta) f(t), \quad u_\theta|_{R=R_0} = U_\theta(\theta) f(t) \tag{1.1}$$

or a stress distribution

$$\sigma_R|_{R=R_0} = F_R(\theta) f(t), \quad \tau_{R\theta}|_{R=R_0} = F_\theta(\theta) f(t). \tag{1.1'}$$

Find the displacement and stress fields in the medium. The problem reduces to solving two wave equations

$$\Delta\varphi = a^2\varphi_{tt}, \ \Delta\boldsymbol{\psi} = b^2\boldsymbol{\varphi}_{tt} \tag{1.2}$$

for the longitudinal φ and transverse $\boldsymbol{\psi}$ potentials under zero initial data

$$\varphi = \varphi_t = 0, \ \boldsymbol{\psi} = \boldsymbol{\psi}_t = 0, \ t = 0 \tag{1.3}$$

and the boundary conditions (1.1) or (1.1)'. In (1.2) $a = 1/V_p$, $b = 1/V_s$, where $V_p = \sqrt{(\lambda + 2\mu)/\rho}$, $V_s = \sqrt{\mu/\rho}$ are the longitudinal and transverse wave velocities, λ, μ are the Lamé parameters, and ρ is the density of the part of the medium in which the source is.

Because of the assumption made about symmetry, the unknown functions are independent of the angle χ and the displacement $u_\chi = 0$; then the transverse potential has the form $\boldsymbol{\psi} = (0, 0, \psi)$ and there are not tangential components of the stress tensor $\tau_{R\chi} = \tau_{\theta\chi} \equiv 0$ (consequently, u_χ and $\tau_{R\chi}$ did not figure in (1.1), and ψ can be written in place of $\boldsymbol{\psi}$ in (1.2) and (1.3)).

Separation of variables permits obtaining expressions for the potentials in the form

$$\overline{\varphi}(R, \theta, s) = \overline{f}(s) \sum_{n=0}^{\infty} \left[A_n(s) \frac{K_\nu(asR)}{\sqrt{R}} + C_n(s) \frac{I_\nu(asR)}{\sqrt{R}} \right] P_n(\cos\theta), \tag{1.4}$$

$$\overline{\psi}(R, \theta, s) = \overline{f}(s) \sum_{n=0}^{\infty} \left[B_n(s) \frac{K_\nu(bsR)}{\sqrt{R}} + D_n(s) \frac{I_\nu(bsR)}{\sqrt{R}} \right] P_n^1(\cos\theta), \tag{1.4'}$$

where the bar above the functions denotes the Laplace transform, s is the Laplace transfor-

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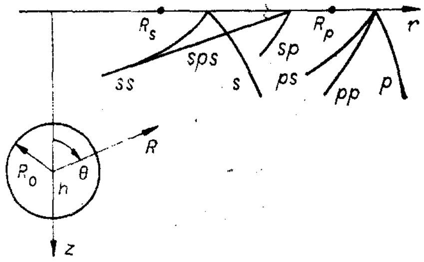

Fig. 1

mation parameter, K_ν is the Macdonald function, $\nu = n + 1/2$; I_ν is the modified Bessel function, P_n is the Legendre polynomial, P_n^1 is the associated Legendre polynomial. Since $I_\nu(y) \to \infty$ as $y \to \infty$, and the initial data are zero, then $C_n(s) = D_n(s) = 0$, $n \geqslant 0$.

A function given on a sphere can be expanded in a series of Legendre polynomials, for instance

$$F_R(\theta) = \sum_{n=0}^{\infty} a_n P_n(\cos\theta), \quad F_\theta(\theta) = \sum_{n=0}^{\infty} b_n P_n^1(\cos\theta). \tag{1.5}$$

There are analogous representations for U_R and U_θ. Now, using (1.4) amd (1.4)' to evaluate the displacements and stresses for $R = R_0$ by known formulas, and comparing them with expansions of the type (1.5), we find the unknown function $A_n(s)$ and $B_n(s)$ and then by using the Mellin inversion formula, we obtain an expression for $\varphi = \varphi(R, \theta, t)$ and for $\psi = \psi(R, \theta, t)$.

The solution found is valid up to emergence of the initiated waves on the boundary $z = 0$. In order to describe the wave reflection-refraction process, expressions must be obtained for the potentials φ and ψ in the cylindrical coordinate system. The formula

$$\sqrt{\frac{2as}{\pi}} \frac{K_{0,5}(asR)}{\sqrt{R}} P_0(\cos\theta) = \frac{e^{-asR}}{R} = \int_0^\infty kJ_0(kr) \frac{e^{-|h-z|\rho_a}}{\rho_a} dk$$

was used in [1], where $\rho_a = \sqrt{k^2 + a^2s^2}$; $R^2 = r^2 + (h-z)^2$; J_0 is the Bessel function. Let us derive an analogous formula for the separate component from (1.4)

$$\frac{K_\nu(asR)}{\sqrt{R}} P_n(\cos\theta) = \sqrt{\frac{\pi}{2as}} \int_0^\infty kJ_0(kr) \frac{H_n}{\rho_a} e^{-|h-z|\rho_a} dk. \tag{1.6}$$

Differentiating both sides of the equality (1.6) with respect to z (this can be done because of the absolute convergence of the integral in the right side of (1.6), and its arbitrary derivative in all space excepting the plane $z = h$), and performing simple manipulations, we obtain a recursion formula for H_n

$$H_{n+1} = \frac{2n+1}{n+1} \frac{\rho_a}{as} H_n - \frac{n}{n+1} H_{n-1}, \quad H_0 = 1, \quad H_1 = \rho_a/as. \tag{1.7}$$

The equality

$$\frac{K_\nu(bsR)}{\sqrt{R}} P_n^1(\cos\theta) = \frac{1}{bs} \sqrt{\frac{\pi}{2bs}} \int_0^\infty k^2 J_1(kr) \frac{Q_n}{\rho_b} e^{-|h-z|\rho_b} dk \tag{1.8}$$

holds for the components from (1.4)'. The functions Q_n are connected by the relationship

$$Q_{n+1} = \frac{2n+1}{n} \frac{\rho_b}{bs} Q_n - \frac{n+1}{n} Q_{n-1}, \quad Q_1 = 1, \quad Q_2 = \rho_b/bs. \tag{1.9}$$

Let us note that (1.7) and (1.9) are valid for $P_n(x)$ and $P_n^1(x)$ (with the replacement of ρ_a/as and ρ_b/bs by x).

2. Let us calculate the displacements on the free surface of the half-space, which occur because of the action of the "thrust" vibration source [2]. In this case

$$f(t) = \sin 2\pi\omega t \varepsilon(t), \quad \varepsilon(t) = \begin{cases} 0, & t<0, \\ 1, & t>0, \end{cases}$$

$$F_R(\theta) = \sigma_0(1+\cos 2\theta) = \sigma_0\left[\frac{2}{3}P_0(\cos\theta) + \frac{4}{3}P_2(\cos\theta)\right], \quad F_\theta(\theta) = 0.$$

Then $a_0 = (2/3)\sigma_0$, $a_1 = 0$, $a_2 = (4/3)\sigma_0$, $a_n = 0$, $n \geqslant 3$ in (1.5). Because of the linearity, the problem dissociates into two with different boundary conditions:

a) $F_R(\theta) = \sigma_0 P_0(\cos\theta) = \sigma_0$; б) $F_R(\theta) = \sigma_0 P_2(\cos\theta)$.

The first problem is solved in [1], and we write down the solution of the second at once, omitting the intermediate calculations.

Upon incidence of a longitudinal wave (see Fig. 1, where p is the incident longitudinal wave, pp, ps, and R_p are the reflected longitudinal, transverse, and Rayleigh waves excited by the p wave)

$$u(t) = K\int_{t_p}^{t}\frac{\Phi(t-\tau)}{\tau V_s}\left[\int_0^\infty J_1(k\xi)\left\{\frac{1}{2\pi i}\int_{(l)}\frac{2\beta g_1}{\zeta^2 R(\zeta)}e^{k(\zeta-\eta\alpha)}d\zeta\right\}dk\right]d\tau, \tag{2.1}$$

$$w(t) = -K\int_{t_p}^{t}\frac{\Phi(t-\tau)}{\tau V_s}\left[\int_0^\infty J_0(k\xi)\left\{\frac{1}{2\pi i}\int_{(l)}\frac{g g_1}{\zeta^2 R(\zeta)}e^{k(\zeta-\eta\alpha)}d\zeta\right\}dk\right]d\tau,$$

$$\Phi(t) = \frac{1}{2\pi i}\int_{(l)}\frac{s\omega_0}{s^2+\omega_0^2}\,\frac{s^4+5s^3+21s^2+48s+48}{d(s)}\,e^{s(t-\gamma)}ds,$$

where $d(s) = \gamma^2 s^6 + \gamma(3+5\gamma+4\gamma^2)s^5 + (3+15\gamma+49\gamma^2-4\gamma^3)s^4 + (15+135\gamma+68\gamma^2-36\gamma^3)s^3 + (135+216\gamma+36\gamma^2-96\gamma^3)s^2 + (1+\gamma)(216-96\gamma^2)s + 216 - 96\gamma^2$, u, w are the displacements in the r and z direction, $t_p = a(\sqrt{r^2+h^2}-R_0)$ is the time of arrival at the point (r, ψ, h) of the longitudinal wave; $K = \sigma_0 R_0^2/\mu$; $\xi = rb/\tau$; $\eta = hb/\tau$; $\zeta = sb/k$; $\alpha = \sqrt{1+\gamma^2\zeta^2}$; $\beta = \sqrt{1+\zeta^2}$; $\gamma = a/b$; $\omega_0 = 2\pi\omega b R_0$; $g_1 = 3+2\gamma^2\zeta^2$; $g = 2+\zeta^2$; $R(\zeta) = g^2 - 4\alpha\beta$; l is the contour in the complex ζ plane that passes to the right of the imaginary axis and parallel to it, and branches of the radicals are determined by the condition $\arg\alpha = \arg\beta = 0$ for $\zeta > 0$.

Upon incidence of a transverse wave (see Fig. 1, where s is an incident transverse wave, and sp, ss, and R_s are the reflected longitudinal, transverse, and Rayleigh waves excited by the s waves, and sps is a conical wave that occurs because of reflection of an sp wave from the free surface)

$$u(t) = K\int_{t_*}^{t}\frac{\Psi(t-\tau)}{\tau V_s}\left[\int_0^\infty J_1(k\xi)\left\{\frac{1}{2\pi i}\int_{(l)}V(\zeta)e^{k(\zeta-\eta\beta)}d\zeta\right\}dk\right]d\tau, \tag{2.2}$$

$$w(t) = -K\int_{t_*}^{t}\frac{\Psi(t-\tau)}{\tau V_s}\left[\int_0^\infty J_0(k\xi)\left\{\frac{1}{2\pi i}\int_{(l)}W(\zeta)e^{k(\zeta-\eta\beta)}d\zeta\right\}dk\right]d\tau,$$

$$\Psi(t) = \frac{1}{2\pi i}\int_{(l)}\frac{s\omega_0}{s^2+\omega_0^2}\,\frac{2(\gamma^3 s^3+5\gamma^2 s^2+12\gamma s+12)}{d(s)}\,e^{s(t-1)}ds,$$

$$t_* = \begin{cases} t_s, & \theta<\theta_*, \\ t_{sp}, & \theta>\theta_*, \end{cases}$$

where $\theta_* = \arcsin\gamma$ is the critical angle of incidence of the transverse wave, $t_s = b(\sqrt{r^2+h^2} - R_0)$, $t_{sp} = b(\gamma r + h\sqrt{1-\gamma^2} - R_0)$ is the time of arrival of the transverse and sp-waves, respectively, at the point under investigation:

$$V(\zeta) = \begin{cases} \dfrac{6\beta g}{\zeta^2 R(\zeta)} - \dfrac{3\beta}{\zeta^4}, & t_{sp}<\tau<t_s, \\ \dfrac{6\beta g}{\zeta^2 R(\zeta)}, & t_s<\tau; \end{cases}$$

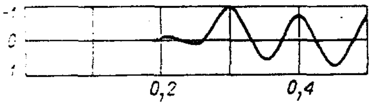

Fig. 2

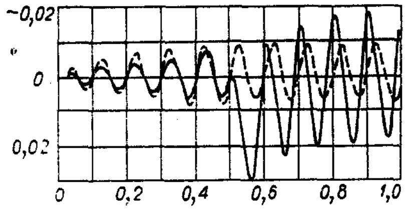

Fig. 3

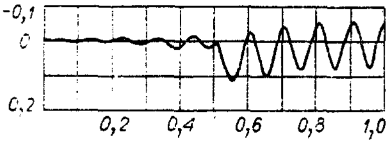

Fig. 4

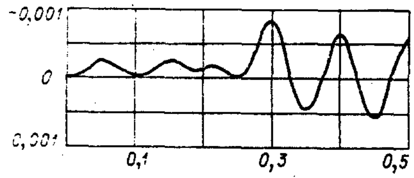

Fig. 5

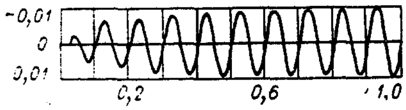

Fig. 6

$$W(\zeta)=\begin{cases}\dfrac{12\alpha\beta}{\zeta^2R(\zeta)}-\dfrac{3}{\zeta^4}, & t_{sp}<\tau<t_s,\\[2mm] \dfrac{12\alpha\beta}{\zeta^2R(\zeta)}, & t_s<\tau.\end{cases}$$

The expressions in the square brackets are calculated by the method of reduction to real integrals described in [3], say. The solution of the initial problem is a linear combination of problems "a" and "b".

The following parameters were selected for the computations: ω = 10 Hz, V_S = 4000 m/ sec, h = 100 m, R_0 = 1 m, $\gamma = 1/\sqrt{3}$, and μ remains arbitrary. The time in seconds is plotted along the abscissa, and the displacement referred to 0.01°K along the ordinate. Results of the computations are represented in Figs. 2 and 3 (solid lines) and 4. Represented in Figs. 5, 3 (dashed line) and 6 are oscillograms of the displacement under the effect of a source of expansion-compression type with the same energy as the "thrust" source.

3. The integrals in the square brackets in (2.1) and (2.2) are extremely complex to evaluate by the method of reduction to real integrals in the case of multilayered media; consequently, different asymptotic methods are used.

The characteristic integral obtained because of the calculation of the displacement field, say, can be written in the form

$$I_j(t)=\int\limits_0^\infty kJ_j(kr)\left\{\frac{1}{2\pi i}\int\limits_{(l)}L(s)\bar{f}(s)M(k,s)\,e^{st-q(k,s)}ds\right\}dk, \tag{3.1}$$

where the function L(s) is known from the solution of the boundary value problem, M(k, s) and q(k, s) are homogeneous functions of zeroth and first order, respectively, corresponding to a certain wave, and j = 0, 1. Let us rewrite (3.1) by using the convolution integral

$$I_j(t)=\int\limits_{t_0}^{t}\left\{\frac{1}{2\pi i}\int\limits_{(l)}sL(s)\,e^{s(t-\tau)}ds\right\}T_j(\tau)\,d\tau,$$

$$T_j(t) = \int_0^\infty kJ_j(kr)\left\{\frac{1}{2\pi i}\int_{(l)} \bar{f}(s)\frac{M(k,s)}{s}e^{st-q(k,s)}ds\right\}dk,$$

t_0 is the time of arrival of a given wave at the point under investigation. Let us make a change of variable $\zeta = sb/k$ in T_j; then

$$T_j(t) = \int_0^\infty kJ_j(kr)\left\{\frac{1}{2\pi i}\int_{(l)} \bar{f}\left(\frac{\zeta k}{b}\right)\frac{M(\zeta)}{\zeta}e^{k\left[\zeta\frac{t}{b}-q(\zeta)\right]}d\zeta\right\}dk.$$

Now, by calculating the inner integral by the method of stationary phase [4] for $f(t) = e^{i\omega t}\varepsilon(t)$ and then integrating with respect to k, we obtain the asymptotic representation of T_j in the neighborhood of the wave front corresponding to the functions q and M

$$T_j(t) \approx -\frac{0.5\pi E_j e^{i\omega\tau_0} + G_j\left[e^{i\omega\tau_0}\mathrm{ci}\,\omega\tau_0 - ie^{-i\omega\tau_0}(0,5\pi + \mathrm{si}\,\omega\tau_0)\right]}{\pi\omega\sqrt{r\,|q''(\zeta_0)|}}, \qquad (3.2)$$

where $tV_s = q'(\zeta_0)$; $\tau_0 = t - t_0$; $E_0 = G_1 = \mathrm{Im}\,[M(\zeta_0)/\zeta_0]$; $E_1 = G_0 = \mathrm{Re}\,[M(\zeta_0)/\zeta_0]$; and si and ci are the integral sines and cosines.

In conclusion, we note the following.

1. Formulas (1.6)-(1.9) permit writing the potentials calculated in a spherical system in a cylindrical coordinate system, thereby affording a possibility of solving problems on the action of sources with inhomogeneous boundary conditions in laminar media with plane-parallel interfaces.

2. Computations of displacements on a half-space surface showed:

a) Under the action of a "thrust" source the amplitude of the displacement w is considerably greater than the amplitude u at distances comparable to the depth h;

b) the build-up, i.e., agreement between the frequency of vibration of the point under consideration with the frequency of the boundary load, occurs from the time ~$15t_s$;

c) the amplitude of the displacement w under the action of a "thrust" source is 2-3 orders greater for small angles of wave incidence than under the action of a compression-expansion type source with the same energy (Figs. 2 and 5, $\theta = 0$), which shows the advantage of the "thrust" source; as the angle θ increases, the difference in amplitudes decreases (Figs. 4 and 6, $\theta = 60°$);

d) the amplitudes of the displacement u differ insignificantly for the afore-mentioned types of sources (Figs. 3, solid and dashed lines, $\theta = 60°$).

3. Asymptotic formulas of the type (3.2) can be used in the neighborhood of wave fronts to calculate displacement and stress fields in multilayered media under the effect of a vibration source.

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